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Mean Dimension of Function Classes with Lebesgue Measurable Spectral Sets

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The notion of mean dimension was introduced in the 1970s by Tikhomirov. It determines the mean number of linear dimensions required to identify an element of a given function class. Tikhomirov then posed the following problem: find the mean dimension of the unit ball B_E^p of the space of L^p-functions on R^n with spectra inside a given Lebesgue measurable bounded set E. In the language of signal analysis: determine the amount of linear information carried by generalized band-limited signals. In this paper Tikhomirov's conjecture on mean dimension is confirmed in certain important cases and yet shown to fail in certain other cases.

1. INTRODUCTION

The mean dimension is one of the averaged characteristics of function classes on the Euclidean space R^n which arises in approximation theory. It was introduced in the 1970s by Tikhomirov and studied in [8, 9, 14, 24, 26–28, 32, 40, 41].

The idea of considering averaged characteristics of function classes goes back to works of Shannon [35, 36] and Kolmogorov and Tikhomirov [16]. They investigated the entropy-like characteristics of classes of random processes on R^1 (Shannon) and of classes of entire functions (Kolmogorov and Tikhomirov). The notion of mean dimension is close to that of the mean entropy. Both of them constitute the mean amount of information which is necessary to identify an element of the given function class. The mean dimension shows how economically we can approximate function classes by finite-dimensional linear subspaces.

Consider a Banach space X and its compact subset A. For every positive integer m, let $L(m)$ be the collection of all finite-dimensional subspaces L of X such that $\dim(L) \leqslant m$. By definition, the Kolmogorov m-widths $\{d_m(A, X)\}$ of the set A are

$$d_m(A, X) = \inf_{L \in L(m)} \sup_{x \in A} \inf_{y \in L} \|x-y\|_X, \qquad m \geqslant 0$$

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(see [31]). The value at $\varepsilon > 0$ of the inverse function, given by

$$K_\varepsilon(A) = \min\{m : d_m(A, X) \leqslant \varepsilon\},$$

is called the ε-dimension of A. It is easy to see that

$$d_m(A, X) \to 0, \qquad m \to \infty,$$

and

$$K_\varepsilon(A) \to \infty, \qquad \varepsilon \to 0.$$

In many problems of approximation theory on R^n we encounter function classes A which are not compact in a given Lebesgue space $L^p(R^n)$, $1 \leqslant p \leqslant \infty$, but have the following property: their restrictions A_t to the cubes

$$C_t = \{x = (x_1, ..., x_n) \in R^n : |x_i| \leqslant t,\ 1 \leqslant i \leqslant n\}, \qquad t > 0, \tag{1}$$

are compact in $L^p(C_t)$. In this case we consider the function of two variables

$$(t, \varepsilon) \to K_\varepsilon(A_t; L^p(A_t))$$

and study its asymptotic behavior as $t \to \infty$, $\varepsilon \to 0$.

By definition, the quantities

$$\bar{K}_\varepsilon^{(u)}(A) = \limsup_{t \to \infty} (2t)^{-n} K_\varepsilon(A_t; L^p(A_t)) \tag{2}$$

and

$$\bar{K}_\varepsilon^{(l)}(A) = \liminf_{t \to \infty} (2t)^{-n} K_\varepsilon(A_t; L^p(A_t))$$

are called the mean upper ε-dimension and the mean lower ε-dimension of the class A, respectively. If the ordinary limit exists in (2), then its value is called the mean ε-dimension of A and denoted by $\bar{K}_\varepsilon(A)$. Finally, if the mean ε-dimension $\bar{K}_\varepsilon(A)$ exists for $0 < \varepsilon < \varepsilon_0$, then the limit

$$\bar{K}(A) = \lim_{\varepsilon \to 0} \bar{K}_\varepsilon(A)$$

is called the mean dimension of A.

Suppose a bounded Lebesgue measurable subset E of R^n is given. Consider the following function class on R^n,

$$B_E^p = \{f \in L^p(R^n) : \operatorname{supp} F(f) \subset E;\ \|f\|_p \leqslant 1\}, \qquad 1 \leqslant p \leqslant \infty. \tag{3}$$

In this definition, F denotes the generalized Fourier transform acting on the Schwartz space $S'(R^n)$ of tempered distributions, and $\operatorname{supp} F(f)$ denotes the support of the distribution $F(f)$. For $f \in L^1(R^n)$ we have

$$F(f)(x) = (2\pi)^{-n/2} \int_{R^n} f(u)\, e^{-ixu}\, du.$$

The set E is called the spectral set of the class B_E^p. By the celebrated Paley–Wiener theorem (see [17]) the class $B^2_{[-a,a]}$, $a > 0$, $n = 1$, coincides with the unit ball in the Paley–Wiener space consisting of restrictions to the real line of entire functions of exponential type a on the complex plane C, which belong to $L^2(R^1)$ on R^1.

Tikhomirov proved that

$$\bar{K}_\varepsilon(B^\infty_{[-a,a]}) = \pi^{-1} a, \qquad 0 < \varepsilon < 1,\ a > 0,\ n = 1$$

(see [41] for more information; in [16, 38] there is a similar formula for the mean entropy). Based on this formula, he posed the following problem: to prove (or to disprove) that for all spectral sets E one has

$$\bar{K}(B_E^p) = (2\pi)^{-n}\, m(E), \qquad 1 \leqslant p \leqslant \infty. \tag{4}$$

In this equality m denotes the n-dimensional Lebesgue measure over R^n. When formula (4) holds it is called Tikhomirov's formula for the class B_E^p.

The dimensionality problem for the spaces of band-limited functions originates in classical papers of Landau, Pollak, and Slepian (see Landau's expository paper [22] for more details). The quantity $(2\pi)^{-n} m(E)$ plays an important role in the theory of sampling and interpolation for the spaces of band-limited functions (see [20]) and coincides with the Nyquist rate $\pi^{-1}a$ in the case $E = [-a, a]$, $a > 0$, $n = 1$.

It is well known that the formula

$$d_m(B_E^2[-t,t]^n, L^2[-t,t]^n) = (\lambda_{m+1}(t))^{1/2}, \qquad m \geqslant 0,$$

holds for the m-widths, where $\lambda_m(t)$ denote the eigenvalues of time and frequency limiting operators

$$T_t(f)(x) = (2\pi)^{-n/2} \int_{[-t,t]^n} f(u)\, F(\chi_E)(u - x)\, du, \qquad t > 0$$

(see [31, Proposition 2.8], where the case $n = 1$, $E = [-a, a]$ is considered; the general case is similar).

It follows that for all spectral sets E we have

$$K_\varepsilon(B_E^2[-t,t]^n, L^2[-t,t]^n) = \min(m : \lambda_{m+1}(t) \leqslant \varepsilon^2).$$

Hence, we may apply Landau's theorem on the asymptotics of the distribution function of the eigenvalues $\lambda_m(t)$ (see [21, Theorem 1]) in our case and get the formula

$$\bar{K}_\varepsilon(B_E^2) = (2\pi)^{-n}\, m(E), \qquad 0 < \varepsilon < 1.$$

Therefore, Landau's theorem provides the solution to Tikhomirov's problem in the case $p = 2$.

Din Dung [8] considered the case of Jordan measurable spectral sets E. The Jordan measurability of E means that

$$m(\partial E) = 0,$$

where ∂E is the boundary of E. He proved that for such spectral sets

$$\bar{K}_\varepsilon(B_E^p) = (2\pi)^{-n}\, m(E), \qquad 1 < p < \infty. \tag{5}$$

Formula (5) is true also for $p = 1, p = \infty$ (see [24, 26] and Theorem 3.1 below).

Let us now define a more general version of the mean dimension. Suppose that instead of a constant rate of approximation ε in the definition of mean dimension, we consider a variable rate of approximation given by some positive non-increasing function $\varphi(t)$, $t > 0$. For every function class A on R^n as above we define the mean upper φ-dimension and the mean lower φ-dimension of the class A by

$$\bar{K}_\varphi^{(u)}(A) = \limsup_{t \to \infty}\, (2t)^{-n}\, K_{\varphi(t)}(A_t;\, L^p(A_t))$$

and

$$\bar{K}_\varphi^{(l)}(A) = \liminf_{t \to \infty} (2t)^{-n}\, K_{\varphi(t)}(A_t;\, L^p(A_t)),$$

respectively. The mean φ-dimension $\bar{K}_\varphi(A)$ of the class A is given by

$$\bar{K}_\varphi(A) = \lim_{t \to \infty}\, (2t)^{-n}\, K_{\varphi(t)}(A_t;\, L^p(A_t))$$

when the former limit exists. The ordinary mean ε-dimension corresponds to the function $\varphi(t) = \varepsilon$, $t > 0$.

Tikhomirov's problem can be reformulated for the mean φ-dimension as follows: for which spectral sets E, numbers p, and rates of approximation φ does the formula

$$\bar{K}_\varphi(B_E^p) = (2\pi)^{-n}\, m(E) \tag{6}$$

hold? This is the problem we are dealing with in the present paper.

The paper is organized as follows. In Section 2 some well-known results are presented.

Section 3 is concerned with the case of Jordan measurable spectral sets. In Theorem 3.1 we give a sufficient condition for the validity of formula (6) expressed in terms of the decay rate of the function φ. It is interesting to note that the logarithmic integral

$$I(\varphi)=\int_0^\infty \frac{|\ln \varphi(t)|}{1+t^2}\,dt$$

arises in this setting. The sufficient condition mentioned above is $I(\varphi)<\infty$. This condition occurs often in problems of function theory and harmonic analysis (see [17]).

Kowalski and Stenger (see [18, 19]) obtained the asymptotic formula

$$\lim_{\varepsilon\to\infty} \frac{K_\varepsilon(B^2_{[-a,a]}[-t,t],L^2[-t,t])\ln\ln(\varepsilon)^{-1}}{\ln(\varepsilon)^{-1}}=1, \qquad a>0, t>0.$$

In the proof of this result they used two-sided estimates for the eigenvalues of time and frequency limiting operators. It is worth noting that our Theorem 3.1 follows neither from the estimates used in [18, 19], nor from the results on distribution of the eigenvalues of time and frequency limiting operators obtained by Landau and Widom (see 23]). The known estimates for the eigenvalues are not sharp enough for our purposes.

In Section 4 we prove that Tikhomirov's formula (4) is not true in the case $1\leqslant p<2$ for some closed spectral sets. The proof uses the notion of the set of uniqueness for the Carleman singularity.

In Section 5 some more counterexamples to Tikhomirov's problem in the case $2<p\leqslant\infty$ are given. It is proved (see Theorem 5.1) that formula (4) fails to be true for some spectral sets of Lebesgue measure zero. It follows that formula (4) does not hold for some open spectral sets (see Corollary 5.2).

Section 6 is concerned with the cases $p=2$ and $p=\infty$. We prove that in these cases formula (6) with $I(\phi)<\infty$ holds for all closed spectral sets (Theorems 6.1 and 6.2). It is not known whether formula (6) with $I(\phi)<\infty$ holds in the case $p=2$ for all spectral sets. As for the case $p=\infty$, the probable solution to Tikhomirov's problem is

$$\bar{K}_\varphi(B_E^\infty)=(2\pi)^{-n}\, m(\operatorname{clos} E),$$

where clos E denotes the closure of the spectral set E, but we were not able to prove this.

Suppose $n=1$ and $E=[-\pi, \pi]$. Special examples of functions φ for which

$$\bar{K}_{\varphi}(B^{p}_{[-\pi,\pi]})=1, \qquad 1\leqslant p\leqslant\infty, \tag{7}$$

are

$$\varphi(t)=\exp\{-t[\ln(1+t)]^{-1-\tau}\}, \qquad t>0, \tau>0,$$

or

$$\varphi(t)=\exp\{-t\ln(1+t)[\ln\ln(3+t)]^{-1-\tau}\}, \qquad t>0, \tau>0, \text{ etc.}$$

(see Theorem 3.1). In Section 7 we prove that for the function

$$\varphi(t)=\exp\{-(1+\delta)\,t\ln(1+t)\}, \qquad t>0, \delta>0,$$

formula (7) does not hold (Theorem 7.1). It would be interesting to find sharp conditions on the functions φ for which formula (7) holds.

Some of the results given in this paper were announced in [14].

2. Preliminaries

We need some simple properties of Jordan measurable sets. The proofs of the following lemmas are left as an exercise for the reader.

Lemma 2.1. (a) *Suppose a number $\varepsilon>0$ and a compact set $G\subset R^n$ are given. Then there exists a Jordan measurable set G_ε such that $G\subset G_\varepsilon$ and*

$$m(G_\varepsilon\backslash G)\leqslant\varepsilon.$$

(b) *Suppose a number ε and a bounded open set $O\subset R^n$ are given. Then there exists a Jordan measurable set O_ε such that $O_\varepsilon\subset O$ and*

$$m(O\backslash O_\varepsilon)\leqslant\varepsilon.$$

Lemma 2.2. *Suppose a number $\varepsilon>0$ and a bounded Jordan measurable set $H\subset R^n$ are given. Then there exist two sets G_ε and D_ε such that*

(1) $G_\varepsilon\subset H\subset D_\varepsilon$;

(2) $G_\varepsilon=\bigcup_{i=1}^{k}\Delta_i$, $D_\varepsilon=\bigcup_{i=1}^{m}\Theta_i$, *where the families $\{\Delta_i\}$ and $\{\Theta_i\}$ consist of cubes with non-intersecting interiors*;

(3) $m(D_\varepsilon\backslash H)\leqslant\varepsilon$, $m(H\backslash G_\varepsilon)\leqslant\varepsilon$.

We make use of a number of known theorems. The first concerns the widths of finite-dimensional unit balls.

THEOREM 2.3 (Tikhomirov [39]). *Let X be a Banach space with the unit ball B. Then for every $L \in L(m+1)$, $m \geqslant 1$, we have*

$$d_m(B \cap L, X) = 1.$$

This theorem is frequently used when one needs to obtain lower estimates for mean dimension.

We also require the use of Blichfeldt's theorem. This well-known assertion provides an estimate for the number of points inside a given Lebesgue measurable set, which belong to shifts of a given net in R^n (see [4, 6]). We need Blichfeldt's theorem in the following formulation.

Let W denote a net in R^n consisting of all points with integral coordinates. For $r > 0$ and $x \in R^n$ denote by $W(r, x)$ the net $rW + x$.

THEOREM 2.4. *Suppose a number $\varepsilon > 0$ and a bounded Lebesgue measurable set $E \subset R^n$ are given. Then there exists $r_\varepsilon > 0$ and for every $r < r_\varepsilon$ there exists $x_r \in R^n$ such that at least $(m(E) - \varepsilon) r^{-n}$ points of the net $W(r, x_r)$ belong to E.*

Theorem 2.4 can be easily deduced from the equality

$$\lim_{r \to 0} \int_{C_1} \left| r^n \sum_{m \in Z^n} \chi_E(x + mr) - m(E) \right| dx = 0,$$

which is true for all bounded Lebesgue measurable sets E and the unit cube C_1 defined by (1).

Now we turn our attention to some results concerning L^p-functions on R^n having spectra inside a cube C_a.

THEOREM 2.5. (a) *Let $f \in L^p(R^n)$ with $1 \leqslant p < \infty$, and* supp $F(f) \subset C_a$, $a > 0$. *Then*

$$\sum_{m \in Z^n} \left| f\left(\frac{\pi m}{a}\right) \right|^p a^{-n} \leqslant c_p \|f\|_p^p,$$

where $c_p > 0$ depends only on p.

(b) *Let $f \in L^p(R^n)$ with $1 < p < \infty$, and* supp $F(f) \subset C_a$, $a > 0$. *Then*

$$\|f\|_p^p \leqslant d_p \sum_{m \in Z^n} \left| f\left(\frac{\pi m}{a}\right) \right|^p a^{-n},$$

where $d_p > 0$ depends only on p.

(c) *Let $f \in L^1(R^n)$ with* supp $F(f) \subset C_a$, $a > 0$. *Then*

$$c'_\delta \sum_{m \in Z^n} \left| f\left(\frac{\pi m}{a+\delta}\right) \right| (a+\delta)^{-n}$$
$$\leqslant \|f\|_1 \leqslant c_\delta \sum_{m \in Z^n} \left| f\left(\frac{\pi m}{a+\delta}\right) \right| (a+\delta)^{-n}, \qquad \delta > 0,$$

where c_δ and c'_δ depend only on δ.

This theorem is well known. For $n=1$ parts (a) and (b) of Theorem 2.5 are due to Plancherel and Pólya. Part (c) was proved by Wiener (see [5]). The multi-dimensional case is similar.

THEOREM 2.6 (the general sampling theorem). *Let $f \in L^p(R^n)$ with $1 \leqslant p \leqslant \infty$, and* supp $F(f) \subset C_a$, $a > 0$. *Then for every $\delta > 0$ and every C^∞-function g such that $g(x) = 1$ for $x \in C_a$, and* supp $g \subset C_{a+\delta}$ *we have*

$$f(x) = \pi^n (a+\delta)^{-n} \sum_{k \in Z^n} f\left(\frac{\pi k}{a+\delta}\right) Fg\left(\frac{\pi k}{a+\delta} - x\right), \qquad x \in R^n, \tag{8}$$

where the series converges uniformly on compact subsets of R^n.

The proof of Theorem 2.6 in the case $n=1$ can be found in [29]. The case $n>1$ is similar (see also [30]).

In [16], Tikhomirov used the Cartwright sampling formula

$$f(x) = \frac{\pi}{\delta(a+\delta)} \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{a+\delta}\right)\left(x - \frac{\pi k}{a+\delta}\right)^{-2}$$
$$\times \sin\left((a+\delta)\left(x - \frac{\pi k}{a+\delta}\right)\right) \sin\left(\delta\left(x - \frac{\pi k}{a+\delta}\right)\right),$$
$$f \in L^p(R^1),\ 1 \leqslant p \leqslant \infty,\ \operatorname{supp} Ff \subset [-a, a],\ a > 0,\ \delta > 0,$$

which is a special case of formula (8).

Remark 2.7. All the definitions and results of this paper can be translated from the language of approximation theory and the theory of entire functions into the language of signal analysis. Traces on R^1 of entire functions of exponential type correspond to band-limited signals. The mean dimension can be interpreted as the mean amount of linear information contained in a signal. It is well known how important sampling formulas are in signal analysis (see [10, 11], where different sampling formulas have been obtained).

3. MEAN φ-DIMENSION AND JORDAN MEASURABLE SPECTRAL SETS

In this section the classes B_E^p with Jordan measurable spectral sets are considered. We prove that formula (6) holds for functions φ with convergent logarithmic integral.

THEOREM 3.1. *Let φ be a positive non-increasing function on $(0, \infty)$ for which $\varphi(t) < 1$, $t > 0$. Assume that*

$$\int_0^\infty \frac{|\ln \varphi(t)|}{1+t^2}\, dt < \infty. \tag{9}$$

Then the equality

$$\bar{K}_\varphi(B_E^p) = (2\pi)^{-n}\, m(E), \qquad 1 \leqslant p \leqslant \infty, \tag{10}$$

holds for all Jordan measurable spectral sets E.

Remark 3.2. The condition imposed on φ in Theorem 3.1 is equivalent to the condition

$$\varphi(t) = \exp\{-\Omega(t)\}, \qquad t > 0, \tag{11}$$

where Ω is a positive non-decreasing function defined on $(0, \infty)$ for which

$$\int_0^\infty \frac{\Omega(t)}{1+t^2}\, dt < \infty. \tag{12}$$

This form of the condition under consideration will be more convenient for us in the proof of Theorem 3.1.

Remark 3.3. It can be easily checked that Theorem 3.1 does not hold without additional restrictions on the function φ. This follows from the equality

$$\lim_{\varepsilon \to 0} K_\varepsilon(B_E^p(C_t); L^p(C_t)) = \infty, \qquad t > 0.$$

Proof of Theorem 3.1. We need in the sequel sharp estimates for the rate of decay of the Fourier transform of a C_0^∞-function. The most general estimates of this type have been obtained by Beurling [2] and Korevaar and Luxemburg [25]. Beurling's results constitute the core of the theory of ultra-distributions (see [2, 3, 34]). We give a short survey of some results from [2, 25].

Let ω be a non-negative function on R^n, continuous at the origin and such that

$$\omega(0)=0 \qquad \text{and} \qquad \omega(\xi+\eta)\leqslant\omega(\xi)+\omega(\eta), \qquad \xi, \eta\in R^n. \tag{13}$$

For example, if Ω is an increasing continuous concave function on $[0, \infty)$ and if $\Omega(0)=0$, then the function $\omega(\xi)=\Omega(|\xi|)$, $\xi\in R^n$, satisfies (13).

For every ω satisfying (13), define a class $D_\omega(R^n)$ as follows. The class consists of all functions $g\in L^1(R^n)$ with compact support such that

$$\int_{R^n} |Fg(\xi)| \exp\{\lambda\omega(\xi)\}\, d\xi<\infty$$

for all $\lambda>0$. Functions $g\in D_\omega(R^n)$ are called test functions.

THEOREM 3.4 (Beurling [2]; see also [3]). *Let ω satisfy* (13). *Then the following conditions are equivalent.*

$$\text{(a)} \qquad \int_{R^n} \frac{\omega(\xi)}{1+|\xi|^{n+1}}\, d\xi<\infty; \tag{14}$$

(b) *For each compact K in R^n and each neighborhood V of K there exists $g\in D_\omega(R^n)$ such that $g(x)=1$, $x\in K$, with* $\operatorname{supp}(g)\subset V$, *and* $0\leqslant g(x)\leqslant 1$ *everywhere*;

(c) *$D_\omega(R^n)$ is non-trivial.*

Let M_c denote the set consisting of all continuous non-negative functions ω on R^n, satisfying the following conditions:

(1) $\omega(\xi)=\Omega(|\xi|)$, where Ω is an increasing continuous concave function on $[0, \infty)$ and $\Omega(0)=0$;

(2) ω satisfies (14);

(3) $\omega(\xi)\geqslant\alpha_1+\alpha_2\ln(1+|\xi|)$, for $\xi\in R^n$, some real number α_1, and positive number α_2.

It follows from Proposition 1.8.6 in [3] that if $\omega\in M_c$, then for all $g\in D_\omega(R^n)$ we have

$$|Fg(\xi)|\leqslant\alpha_{g,\lambda}\exp\{-\lambda\omega(\xi)\}, \qquad \xi\in R^n, \lambda>0. \tag{15}$$

In [25] Korevaar and Luxemburg obtained a stronger result in the one-dimensional case. They showed that for every $a>0$ and for every non-negative, non-increasing function ω satisfying

$$\int_0^\infty \frac{\omega(x)}{1+x^2}\, dx<\infty$$

there exists an entire function $f(z)$, $z=x+iy\in C$, of exponential type $<a$ such that

$$|f(x)|\leqslant \exp\{-\omega(|x|)\}, \qquad x\in R^1.$$

If ω satisfies condition 3 above then it can be easily seen that Ff is a C_0^∞-function such that $\operatorname{supp}(Ff)\subset[-a,a]$.

Let us denote by M the class of functions ω on R^n satisfying the following conditions:

(a) ω is representable in the form

$$\omega(\xi)=\Omega(|\xi|), \qquad \xi\in R^n,$$

where Ω is a non-negative, non-decreasing function on $[0,\infty)$;

(b) ω satisfies condition 3 above;

(c) ω satisfies inequality (14).

We prove the following simple generalization of results from [2, 25].

LEMMA 3.5. *If $\omega\in M$, then there exists a function $g\in C_0^\infty$ such that $\int g\,dx>0$ and the estimate* (15) *holds for g and ω.*

Proof. In [25] Korevaar and Luxemburg considered a function

$$f(z)=c\prod_{k=1}^{\infty}\cos\varepsilon_k z, \qquad z\in C.$$

They proved that for some choice of a positive constant c and of a positive sequence $\{\varepsilon_k\}$ the function f is entire and has exponential type $\leqslant a$. Moreover, the estimate

$$|f(x)|\leqslant \exp\{-\omega(|x|)\}, \qquad x\in R^1.$$

holds.

Assumption 3 for the function ω allows us to prove that

$$f(x)=Fg(x), \qquad g\in C_0^\infty.$$

The estimate (15) now follows from the simple fact that for every $\omega\in M$ there exists an increasing function $\tilde\omega\in M$ that majorizes ω and satisfies

$$\lim_{x\to\infty}\frac{\tilde\omega(x)}{\omega(x)}=\infty$$

(see Lemma 3.7 below, where a stronger result is obtained). It is clear from the definition of g that $\int g\,dx > 0$.

In the case $n \geqslant 2$, Lemma 3.5 follows from the one-dimensional case by the following reasoning. Suppose $\omega \in M$. Then the function

$$\tau(x) = \Omega(n^{1/2}\,|x|), \qquad x \in R^1,$$

belongs to the class M for $n = 1$. By the one-dimensional result, there exists a function $h \in C_0^\infty(R^1)$ such that

$$|Fh(x)| \leqslant c_\lambda \exp\{-\lambda \tau(x)\}, \qquad \lambda > 0.$$

Consider a new function of n variables

$$g(x) = h(x_1) \cdots h(x_n), \qquad x = (x_1, ..., x_n) \in R^n.$$

It is clear that $g \in C_0^\infty(R^n)$ and

$$|Fg(x)| \leqslant \tilde{c}_\lambda \exp\{-\lambda[\tau(x_1) + \cdots + \tau(x_n)]\} \leqslant \tilde{c}_\lambda \exp\{-\lambda\omega(x)\},$$

which proves Lemma 3.5.

We need one more lemma.

LEMMA 3.6. *Let $\omega \in M$. Then for every $a > 0$, $\delta > 0$ there exists a function $g_{a,\delta} \in C_0^\infty$ such that*

(a) $0 \leqslant g_{a,\delta}(x) \leqslant 1$, $x \in R^n$;

(b) $g_{a,\delta}(x) = 1$, $x \in C_a$;

(c) $\operatorname{supp}(g_{a,\delta}) \subset C_{a+\delta}$;

(d) *For every $\lambda > 0$ there exists a positive constant α_λ, depending only on λ, such that*

$$|Fg_{a,\delta}(x)| \leqslant \alpha_\lambda (a+\delta)^n \exp\{-\lambda\omega(\delta\xi)\}, \qquad \xi \in R^n.$$

Proof. It follows from Lemma 3.5 that there exists a function $h \in C_0^\infty$ for which

$$\operatorname{supp}(h) \subset C_{1/2}, \qquad \int h\,dx > 0,$$

and

$$|Fh(\xi)| \leqslant \beta_\lambda \exp\{-\lambda\omega(\xi)\}, \qquad \xi \in R^n, \lambda > 0.$$

Fix such a function h and consider convolutions

$$g_{a,\delta}(x)=\chi * h_\delta(x), \qquad x\in R^n,$$

where χ is the characteristic function of the cube $C_{a+\delta/2}$, and $h_\delta(x)=\delta^{-n}h(\delta^{-1}x)$, with $\delta>0$.

Now it is easy to prove that the function $g_{a,\delta}$ satisfies all the required conditions.

LEMMA 3.7. *Suppose Ω is a non-decreasing positive function on $(0,\infty)$ for which (12) is true. Then there exists an increasing function $\bar{\Omega}$ such that $\Omega(t)\leqslant\bar{\Omega}(t)$, $t>0$, inequality (12) holds for $\bar{\Omega}$, and*

$$\lim_{t\to\infty} t^{-1}\bar{\Omega}^{-1}(\alpha\Omega(t))=0 \tag{16}$$

for all $\alpha>0$.

The inverse function $\bar{\Omega}^{-1}$ in Lemma 3.7 is defined by

$$\bar{\Omega}^{-1}(y)=\inf\{x:\bar{\Omega}(x)>y\}.$$

Proof of Lemma 3.7. For every integer k there exists an increasing sequence $\{\delta_k\}$, $0\leqslant k<\infty$, such that $\delta_0=0$ and

$$\int_{\delta_k}^{\infty}\frac{\Omega((k+1)\,t)}{1+t^2}\,dt\leqslant 2^{-(k+2)}, \qquad k\geqslant 1.$$

It follows that

$$\int_{\delta_k}^{\delta_{k+1}}\frac{2^{k+2}\Omega((k+1)\,t)}{(k+1)^2\,(1+t^2)}\,dt\leqslant(k+1)^{-2}, \qquad k\geqslant 1.$$

Define a new function as

$$\bar{\Omega}(t)=(k+1)^{-2}\,2^{k+2}\Omega((k+1)\,t)$$

for $\delta_k\leqslant|t|<\delta_{k+1}$, $k\geqslant 0$. Then we have

$$\int_0^{\infty}\frac{\bar{\Omega}(t)}{1+t^2}\,dt<\infty,$$

and the function $\bar{\Omega}$ increases and satisfies

$$\Omega(t)\leqslant\bar{\Omega}(t).$$

Suppose (16) does not hold for some α. Then there exists a sequence $\{t_j\}$ with $t_j \to \infty$ as $j \to \infty$, for which

$$t_j^{-1}\bar{\Omega}^{-1}(\alpha\Omega(t_j)) \geqslant \rho > 0, \qquad j \geqslant 1.$$

It follows that

$$\bar{\Omega}(\rho t_j) \leqslant \alpha\Omega(t_j), \qquad j \geqslant 1. \tag{17}$$

On the other hand, if $k(j)$ is such that $\delta_{k(j)} \leqslant \rho t_j < \delta_{k(j)+1}$, $j \geqslant 1$, then $k(j) \to \infty$ as $j \to \infty$ and

$$\bar{\Omega}(\rho t_j) \geqslant (k(j)+1)^{-2}\, 2^{k(j)+2}\Omega(t_j), \qquad j > j_0.$$

The last inequality contradicts (17) and Lemma 3.7 is proved.

Let us proceed with the proof of Theorem 3.1. By monotonicity of the mean dimension with respect to the functions φ we have

$$\bar{K}_{\varphi}^{(l)}(B_E^p) \geqslant \bar{K}_{\varepsilon}(B_E^p), \qquad 0 < \varepsilon < 1.$$

It follows that

$$\bar{K}_{\varphi}^{(l)}(B_E^p) \geqslant (2\pi)^{-n}\, m(E) \tag{18}$$

by Din Dung's theorem (see (5)).

Thus, we need only prove the upper estimate for the mean dimension

$$\bar{K}_{\varphi}^{(u)}(B_E^p) \leqslant (2\pi)^{-n}\, m(E), \tag{19}$$

where $\Omega(t) = |\ln \varphi(t)|$ satisfies all the required conditions and the additional assumption

$$\Omega(t) \geqslant \ln(1+t), \qquad t > 0.$$

If the last inequality does not hold for Ω, then we consider a new function

$$\Theta(t) = \Omega(t) + \ln(1+t),$$

for which all the conditions are satisfied.

Now let us define a function $\omega(t) = \bar{\Omega}(|x|)$, $x \in R^n$, where $\bar{\Omega}$ corresponds to Ω by Lemma 3.7. Then for every $a > 0$, $0 < \delta < a$, there exists a function $g_{a,\delta}$ satisfying conditions (a)–(d) of Lemma 3.6.

Suppose $f \in L^p(R^n)$ with $1 < p < \infty$, and $\operatorname{supp}(Ff) \subset C_a$. By Theorem 2.6 we have

$$f(x) = \pi^n(a+\delta)^{-n} \sum_{k \in Z^n} f\left(\frac{\pi k}{a+\delta}\right) Fg_{a,\delta}\left(\frac{\pi k}{a+\delta} - x\right), \qquad x \in R^n.$$

For the sake of simplicity let us write g instead of $g_{a,\delta}$. Our goal is to obtain an upper estimate for

$$I(N, t)=\int_{C_t}\left|\sum_{k\in A_N}(a+\delta)^{-n} f\left(\frac{\pi k}{a+\delta}\right) Fg\left(\frac{\pi k}{a+\delta}-x\right)\right|^p dx,$$

where $t>0$, $N>\pi^{-1}(a+\delta)\,t$, $A_N=Z^n\backslash C_N$.

Using Hölder's inequality and Theorem 2.5 we get

$$\begin{aligned} I(N, t) &\leqslant \int_{C_t}\sum_{k\in A_N}(a+\delta)^{-np}\left|f\left(\frac{\pi k}{a+\delta}\right)\right|^p \\ &\quad\times\left(\sum_{k\in A_N}\left|Fg\left(\frac{\pi k}{a+\delta}-x\right)\right|^q\right)^{p-1} dx \\ &\leqslant c(a+\delta)^{-n(p-1)}\|f\|_p^p \\ &\quad\times\int_{C_t}\left(\sum_{k\in A_N}\left|Fg\left(\frac{\pi k}{a+\delta}-x\right)\right|^q\right)^{p-1} dx, \end{aligned} \tag{20}$$

where $q=p(p-1)^{-1}$. In the sequel we let c stand for different positive constants which may depend on p, n, and Ω but do not depend on a, t, δ, and f.

Assume $1<p\leqslant 2$. Using Lemma 3.6 with

$$\lambda=p^{-1}(n+1)+1$$

and inequality (20), we obtain

$$I(N, t)\leqslant c\,\|f\|_p^p\, a^n\int_{C_t}\left[\sum_{k\in A_N}\exp\left\{-\lambda q\bar{\Omega}\left(\delta\left|\frac{\pi k}{a+\delta}-x\right|\right)\right\}\right]^{p-1} dx.$$

Moreover as

$$x\in C_t,\qquad \pi k(a+\delta)\notin C_l,\qquad l=\pi N(a+\delta)^{-1},$$

we have

$$\begin{aligned} I(N, t) &\leqslant c\,\|f\|_p^p\, a^n\exp\left\{-p\bar{\Omega}\left(\delta\left|\frac{\pi N}{a+\delta}-t\right|\right)\right\} \\ &\quad\times\int_{C_t}\left(\sum_{k\in Z^n}\exp\left\{-p^{-1}q(n+1)\,\bar{\Omega}\left(\delta\left|\frac{\pi k}{a+\delta}-x\right|\right)\right\}\right)^{p-1} dx \\ &\leqslant c\,\|f\|_p^p\, a^n\exp\left\{-\bar{\Omega}\left(\delta\left|\frac{\pi N}{a+\delta}-t\right|\right)\right\}\times S, \end{aligned} \tag{21}$$

where S denotes the previous integral.

It is easy to see that

$$S=\int_{C_t}\sum_k \exp\left\{-(n+1)\,\bar{\Omega}\left(\delta\left|\frac{\pi k}{a+\delta}-x\right|\right)\right\}dx$$

$$=\sum_k \delta^{-n}\int_{D_k}\exp\{-(n+1)\,\bar{\Omega}(|v|)\}\,dv$$

$$\leqslant c\delta^{-n}\sum_k\int_{D_k}(1+|v|)^{-n-1}\,dv, \tag{22}$$

where

$$D_k=\delta\left(C_t-\frac{\pi k}{a+\delta}\right).$$

It follows from (22) that

$$S\leqslant c\delta^{-n}(a+\delta)^n\,t^n. \tag{23}$$

The foregoing reasoning used the following simple observation: For every $v\in R^n$ there are at most $c(a+\delta)^n\,t^n$ points $k\in Z^n$ such that

$$v\in\delta\left(C_t-\frac{\pi k}{a+\delta}\right).$$

Using (21) and (23), we obtain

$$I(N,t)\leqslant c\,\|f\|_p^p\,a^n(a+\delta)^n\,\delta^{-n}t^n\exp\left\{-\bar{\Omega}\left(\delta\left(\frac{\pi N}{a+\delta}-t\right)\right)\right\},\qquad 1<p\leqslant 2. \tag{24}$$

Suppose a function $f\in B^p_{C_a}$ and numbers $t>0$, $\delta>0$ are given. Choose $N=N(t)$ in (24) so that

$$I(N(t),t)\leqslant\varphi(t)^p.$$

Then

$$K_{\varphi(t)}(B^p_{C_a}(C_t);\,L^p(C_t))\leqslant t^{-n}N(t)^n. \tag{25}$$

It is easy to see that the $(2N(t))^n$-dimensional linear subspace of $L^p(C_t)$ with the basis given by

$$\left\{Fg\left(\frac{\pi k}{a+\delta}-x\right)\right\},\qquad k\in C_{N(t)},\ x\in C_t,$$

approximates the class $B^p_{C_a}[-t,t]^n$ with error $\varphi(t)$.

Now choose $V(t) > 0$ so that

$$ca^n(a+\delta)^n \delta^{-n} t^n \exp\left\{-\bar{\Omega}\left(\delta\left(\frac{\pi V(t)}{a+\delta} - t\right)\right)\right\} = \varphi(t)^p \tag{26}$$

with the constant c which appears in (24). Take $N(t) = [V(t)] + 1$, where $[\tau]$ denotes the integral part of τ. From (26) we obtain

$$N(t) = [\pi^{-1}(a+\delta)(t + \delta^{-1}\bar{\Omega}^{-1}(\ln(c\varphi(t)^{-p} a^n(a+\delta)^n \delta^{-n} t^n)))] + 1, \quad t > t_0.$$

For large values of t, the inequality $\Omega(t) \geqslant \ln(1+t)$ yields

$$\begin{aligned} N(t) &\leqslant \pi^{-1}(a+\delta)\, t + \pi^{-1}\delta^{-1}(a+\delta)\, \bar{\Omega}^{-1}(\ln(\varphi(t)^{-p} t^{n+1})) \\ &\leqslant \pi^{-1}(a+\delta)\, t + \pi^{-1}\delta^{-1}(a+\delta)\, \bar{\Omega}^{-1}(p\Omega(t) + (n+1)\ln t) \\ &\leqslant \pi^{-1}(a+\delta)\, t + \pi^{-1}\delta^{-1}(a+\delta)\, \bar{\Omega}^{-1}((p+n+1)\, \Omega(t)). \end{aligned} \tag{27}$$

Now the estimate

$$\bar{K}_\varphi^{(u)}(B_{C_a}^p) \leqslant \pi^{-n}(a+\delta)^n, \qquad a > 0, \delta > 0, \tag{28}$$

follows from (25), (27), and Lemma 3.7.

Finally, (28) yields the required estimate (19), because the left-hand side of (28) does not depend on δ.

Thus

$$\bar{K}_\varphi(B_{C_a}^p) = (2\pi)^{-n}\, m(C_a), \qquad a > 0, \tag{29}$$

by (18) and (19).

Now let us assume $E = C_a$, $a > 0$, $p > 2$. The proof of Theorem 3.1 in this case proceeds as above but there are some minor differences.

First of all we put

$$\lambda = p^{-1}(n+1)(p-1) + 1$$

instead of

$$\lambda = p^{-1}(n+1) + 1.$$

Using Minkovski's and Hölder's inequalities we obtain the following estimate for the quantity S defined in (21):

$$S^{1/(p-1)} \leqslant \left\{ \int_{C_t} \left(\sum_k \exp\left\{ -(\lambda-1)\, q\bar{\Omega}\left(\delta \left| \frac{\pi k}{a+\delta} - x \right| \right)\right\} \right)^{p-1} dx \right\}^{1/(p-1)}$$

$$\leqslant \sum_k \left\{ \int_{C_t} \exp\left\{ -(\lambda-1)\, p\bar{\Omega}\left(\delta \left| \frac{\pi k}{a+\delta} - x \right| \right)\right\} dx \right\}^{1/(p-1)}$$

$$\leqslant \left\{ \sum_k (|k|+1)(n+1)(p-2) \right.$$

$$\left. \times \int_{C_t} \exp\left\{ -(n+1)(p-1)\, \bar{\Omega}\left(\delta \left| \frac{\pi k}{a+\delta} - x \right| \right)\right\} dx \right\}^{1/(p-1)}$$

$$\times \left\{ \sum_k (|k|+1)^{-(n+1)} \right\}^{(p-2)/(p-1)}$$

Hence

$$S \leqslant c\delta^{-n} \sum_k (|k|+1)^{(n+1)(p-2)} \int_{V_k} \exp\{-(n+1)(p-1)\,\bar{\Omega}(|v|)\}\, dv$$

$$\leqslant c\delta^{-n} \int \exp\{-(n+1)(p-1)\,\bar{\Omega}(|v|)\}\, dv$$

$$\times \sum_k (|k|+1)^{(n+1)(p-2)} \chi_{D_k}(v), \tag{30}$$

where

$$V_k = \delta\left(C_t - \frac{\pi k}{a+\delta} \right).$$

For every $v \in R^n$ there are at most $c(a+\delta)^n t^n$ points $k \in Z^n$ for which $v \in D_k$. Moreover, we have the estimate

$$|k| \leqslant \pi^{-1}(a+\delta)\sqrt{n}\, t + \pi^{-1}\delta^{-1}(a+\delta)\, |v|.$$

for such points.

Now it follows from (30) that

$$S \leqslant c\delta^{-n}(a+\delta)^n t^n \int ((a+\delta)\, t + \delta^{-1}(a+\delta)\, |v|)^{(n+1)(p-2)}$$

$$\times \exp\{-(n+1)(p-2)\,\bar{\Omega}(|v|)\}\, dv$$

$$\leqslant c\delta^{-n}(a+\delta)^n t^n \int (at + a\delta^{-1}\, |v|)^{(n+1)(p-2)}$$

$$\times (1+|v|)^{-(n+1)(p-1)}\, dv$$

$$\leqslant c\delta^{-n}(a+\delta)^n t^n((at)^{(n+1)(p-2)} + (\delta^{-1}a)^{(n+1)(p-2)}).$$

For fixed a and δ we have

$$S \leqslant c(a+\delta)^n \delta^{-n} a^{(n+1)(p-2)} t^{(n+1)(p-2)+n}, \qquad t > t_0.$$

Now we can estimate $I(N, t)$ as in the case $1 < p \leqslant 2$. The new value of $N(t)$ is

$$N(t) = [\pi^{-1}(a+\delta)(t+\delta^{-1}\bar{\Omega}^{-1}(\ln(c\varphi(t)^{-p}(a+\delta)^n \times \delta^{-n} a^{(n+1)(p-2)+n} t^{(n+1)(p-2)+n})))] + 1, \qquad t > t_0.$$

In the end we get for the mean upper dimension the estimate

$$\bar{K}_\varphi^{(u)}(B_{C_a}^p) \leqslant \pi^{-n}(a+\delta)^{-n}, \qquad a > 0, \delta > 0.$$

This completes the proof of (29) for $1 < p < \infty$.

The next step consists in proving (10) for all spectral sets E, which can be represented as finite unions

$$E = \bigcup_{m=1}^{M} \Delta_m, \qquad \text{with} \quad \Delta_m = C_{a_m} + h_m,\ a_m > 0,\ h_m \in R^n, \qquad 1 \leqslant m \leqslant M,$$

of cubes Δ_m with non-intersecting interiors.

For every function $f \in B_E^p$, $1 < p < \infty$, we have

$$f = \sum_{m=1}^{M} f_m, \qquad f_m = F^{-1}(\chi_m F(f)),\ \chi_m = \chi_{\Delta_m},\ 1 \leqslant m \leqslant M.$$

It is well known that the characteristic functions of cubes are the Fourier multipliers in $L^p(R^n)$, $1 < p < \infty$, that is,

$$\|f_m\|_p \leqslant \alpha_p \|f\|_p, \qquad 1 \leqslant m \leqslant M \tag{31}$$

(see [36, Theorem 4, Chap. 4]).

Since $\operatorname{supp} F(f_m) \subset \Delta_m$, we have $\operatorname{supp} F(h_m) \subset C_{a_m}$, where

$$h_m(x) = f_m(x) \exp\{-ih_m x\}, \qquad x \in R^n, 1 \leqslant m \leqslant M.$$

Therefore, by Theorem 2.6 we get

$$f_m(x) = \sum_{k \in Z^n} f_m^{(k)}(x),$$

with

$$f_m^{(k)}(x) = \pi^n \exp\{ih_m x\}(a_m+\delta_m)^{-n} \exp\{-ih_m \pi k(a_m+\delta_m)^{-1}\}$$
$$\times f_m\left(\frac{\pi k}{a_m+\delta_m}\right) Fg_m\left(\frac{\pi k}{a_m+\delta_m} - x\right),$$
$$x \in R^n,\ \delta_m \leqslant a_m,\ 1 \leqslant m \leqslant M,$$

and

$$g_m = g_{a_m, \delta_m}, \qquad 1 \leqslant m \leqslant M.$$

Consider an integral

$$I(N_1, ..., N_m, t)$$
$$= \int_{C_t} \left| \sum_{m=1}^{M} \sum_{k \in A_m} f_m^{(k)}(x) \right|^p dx, \qquad t > 0,\ N_m > \pi(a_m+\delta_m)^{-1} t,$$

where $A_m = Z^n \setminus C_{N_m}$.

Our next goal is to estimate this integral. We use the same ideas as before. First we consider the case $1 < p \leqslant 2$. By Hölder's inequality, Theorem 2.5, and Lemma 3.6 with $\lambda = (n+1)p^{-1} + 1$, we get

$$I(N_1, ..., N_m, t)$$
$$\leqslant \int_{C_t} \left\{ \sum_{m=1}^{M} \sum_{k} \left| f_m\left(\frac{\pi k}{a_m+\delta_m}\right)\right|^p (a_m+\delta_m)^{-n} \right\}$$
$$\times \left\{ \sum_{m=1}^{M} \sum_{k \in A_m} (a_m+\delta_m)^{-n} \left| Fg_m\left(\frac{\pi k}{a_m+\delta_m} - x\right)\right|^q \right\}^{p-1} dx$$
$$\leqslant c \sum_{m=1}^{M} \|f_m\|_p^p \int_{C_t} \left\{ \sum_{m=1}^{M} \sum_{k \in A_m} \exp\left\{-\lambda q \omega\left(\delta_m\left(\frac{\pi k}{a_m+\delta_m} - x\right)\right)\right\}\right.$$
$$\left. \times (a_m+\delta_m)^{n(q-1)} \right\}^{p-1} dx.$$

Let us first consider the simplest case, $p = 2$. It is clear that

$$\sum_{m=1}^{M} \|f_m\|_2^2 = \|f\|_2^2 \leqslant 1. \tag{32}$$

Hence for $p=2$ we have

$$
\begin{aligned}
& I(N_1, \dots, N_m, t) \\
& \quad \leqslant c \int_{C_t} \sum_{m=1}^{M} \exp\left\{-2\bar{\Omega}\left(\delta_m \left|\frac{\pi N_m}{a_m+\delta_m}-t\right|\right)\right\} \\
& \qquad \times \sum_{k \in A_m} \exp\left\{-(n+1)\,\bar{\Omega}\left(\delta_m \left|\frac{\pi k}{a_m+\delta_m}-x\right|\right)\right\}(a_n+\delta_m)^n\,dx \\
& \quad \leqslant c \sum_{m=1}^{M} (a_m+\delta_m)^n \exp\left\{-\bar{\Omega}\left(\delta_m\left(\frac{\pi N_m}{a_m+\delta_m}-t\right)\right)\right\} \\
& \qquad \times \sum_k \delta_m^{-n} \int_{D(m,k)} (1+|v|)^{-n-1}\,dv,
\end{aligned}
$$

where

$$
D(m,k)=\delta_m\left(C_t-\frac{\pi k}{a_m+\delta_m}\right).
$$

Therefore,

$$
\begin{aligned}
& I(N_1, \dots, N_m, t) \\
& \quad \leqslant c \sum_{m=1}^{M} \delta_m^{-n}(a_m+\delta_m)^{2n}\, t^n \exp\left\{-\bar{\Omega}\left(\delta_m\left(\frac{\pi N_m}{a_m+\delta_m}-t\right)\right)\right\}. \qquad (33)
\end{aligned}
$$

Now choose $V_m(t)$, $1 \leqslant m \leqslant M$, so that

$$
c\delta_m^{-n}(a_m+\delta_m)^n\, t^n \exp\left\{-\bar{\Omega}\left(\delta_m\left(\frac{\pi V_m(t)}{a_m+\delta_m}-t\right)\right)\right\}=\varphi(t)^2\,(m(E))^{-1}
$$

and put $N_m(t)=[V_m(t)]+1$, $\delta_m=\tau a_m$, $0<\tau<1$. Then

$$
\begin{aligned}
N_m(t) = {} & [\pi^{-1}a_m(1+\tau)\,t+\pi^{-1}(1+\tau)\,\tau^{-1}\bar{\Omega}^{-1} \\
& \times(\ln(cm(E)(1+\tau)^n\,\tau^{-n}t^n\varphi(t)^{-2})]+1. \qquad (34)
\end{aligned}
$$

Using (33), we obtain

$$
I(N_1, \dots, N_m, t) \leqslant \sum_{m=1}^{M} (a_m+\delta_m)^n\, \varphi(t)^2\,(m(E))^{-1} \leqslant \varphi(t)^2.
$$

Now it is easy to get the estimate

$$
K_{\varphi(t)}(B_E^2(C_t);\, L^2(C_t)) \leqslant \sum_{m=1}^{M} (2N_m(t))^n. \qquad (35)
$$

For big values of t we have from (34)

$$\begin{aligned} N_m(t) &\leqslant \pi^{-1} a_m(1+\tau)\, t + \pi^{-1}(1+\tau)\,\tau^{-1}\bar{\Omega}^{-1}(\ln(\varphi(t)^{-2}\, t^{n+1})) \\ &\leqslant \pi^{-1} a_m(1+\tau)\, t + \pi^{-1}(1+\tau)\,\tau^{-1}\bar{\Omega}^{-1}(2\Omega(t) + (n+1)\ln t) \\ &\leqslant \pi^{-1} a_m(1+\tau)\, t + \pi^{-1}(1+\tau)\,\tau^{-1}\bar{\Omega}^{-1}((n+3)\,\Omega(t)). \end{aligned}$$

Now Lemma 3.7 and the inequality (35) give

$$\bar{K}_\varphi^{(u)}(B_E^2) \leqslant (2\pi)^{-n}\, m(E)(1+\tau)^n, \qquad \tau > 0.$$

It follows that

$$\bar{K}_\varphi^{(u)}(B_E^2) \leqslant (2\pi)^{-n}\, m(E).$$

This proves Theorem 3.1 for $p = 2$ and for spectral sets which are finite unions of cubes.

The case $1 < p < 2$ is similar. The only difference is that in this case the inequality (32) is not true, but we may use (31) instead. This gives us the estimate

$$\begin{aligned} &I(N_1, ..., N_m, t) \\ &\quad \leqslant cM \int_{C_t} \left\{ \sum_{m=1}^{M} (a_m+\delta_m)^n \exp\left\{ -p\bar{\Omega}\left(\delta_m \left| \frac{\pi N_m}{a_m+\delta_m} - t \right| \right) \right\} \right. \\ &\qquad \times \sum_{k \in A_m} \exp\left\{ -(n+1)\,\bar{\Omega}\left(\delta_m \left| \frac{\pi k}{a_m+\delta_m} - x \right| \right) \right\} dx \\ &\quad \leqslant cM \sum_{m=1}^{M} \delta_m^{-n}(a_m+\delta_m)^{2n}\, t^n \exp\left\{ -\Omega\left(\delta_m \left(\frac{\pi N_m}{a_m+\delta_m} - t \right) \right) \right\}. \end{aligned} \tag{36}$$

Now we can easily complete the proof of Theorem 3.1 for $1 < p < 2$ and for spectral sets which are finite unions of cubes, using (33) and (36).

The case $p > 2$ can be treated similarly.

Thus Theorem 3.1 is proved for $1 < p < \infty$ and for all spectral sets which are finite unions of cubes. Now the case of Jordan measurable spectral sets in Theorem 3.1 follows by monotonicity of $\bar{K}_\varphi(B_E^p)$ with respect to inclusions of spectral sets and by Lemma 2.2.

In the remaining cases $p = 1$, $p = \infty$ we do not have the Fourier multiplier theorem (31), but we can use instead the C_0^∞-resolutions of identity $\{\Psi_m\}$, such that

$$\sum_{m=1}^{M} \Psi_m(x) = 1, \qquad x \in E,$$

and

$$\operatorname{supp} \Psi_m \subset C_{a_m+\delta_m} + h_m, \qquad 1 \leqslant m \leqslant M.$$

We have

$$f = \sum_{m=1}^{M} f_m, \qquad f_m = F^{-1}(\Psi_m F(f)), \qquad 1 \leqslant m \leqslant M.$$

It is clear that the functions Ψ_m are the Fourier multipliers in $L^1(R^n)$ and $L^\infty(R^n)$. Now the proof of Theorem 3.1 in the remaining cases can be concluded using the same methods as those above.

Remark 3.8. (1) Assume the spectral set E is bounded and closed. Then by Lemma 2.1 and Theorem 3.1 we have

$$\bar{K}_\varphi^{(u)}(B_E^p) \leqslant (2\pi)^{-n}\, m(E), \qquad 1 \leqslant p \leqslant \infty.$$

(2) Assume the spectral set E is bounded and open. Then by Lemma 2.1 and Theorem 3.1 we have

$$\bar{K}_\varphi^{(l)}(B_E^p) \geqslant (2\pi)^{-n}\, m(E), \qquad 1 \leqslant p \leqslant \infty.$$

4. The Case $1 \leqslant p < 2$. Counterexamples to Tikhomirov's Formula

In this section we prove that Tikhomirov's formulas (5) and (6) do not hold in the case $1 \leqslant p < 2$ for some closed spectral sets. This result can be easily deduced from the existence of closed sets of uniqueness for the Carleman singularity.

Definition 4.1. A function $f \in L^1[-\pi, \pi]^n$ has the Carleman singularity if its Fourier coefficients $c_m(f)$, $m \in Z^n$, satisfy

$$\sum_m |c_m(f)|^p = \infty, \qquad 0 < p < 2.$$

Definition 4.2. A closed set $E \subset [-\pi, \pi]^n$ is called the set of uniqueness for the Carleman singularity if every function $f \in L^1[-\pi, \pi]^n$ such that $f(x) = 0$, $x \in [-\pi, \pi]^n \backslash E$, has the Carleman singularity.

Katznelson proved in [15] that for $n = 1$ and for every ε, $0 < \varepsilon < 2\pi$, there exists a set of uniqueness E_ε for the Carleman singularity satisfying

$$m([-\pi, \pi] \backslash E_\varepsilon) \leqslant \varepsilon.$$

A similar result is true for all complete orthonormal systems in $L^2[0, 1]$ (see [7, 12]; more general results can be found in [13]). Since the interval $[-\pi, \pi]$ with the normalized Lebesgue measure and the torus $[-\pi, \pi]^n$ with the normalized n-dimensional Lebesgue measure are isomorphic as measure spaces, we easily deduce that there exist sets of uniqueness for the Carleman singularity with respect to the multiple trigonometric system on $[-\pi, \pi]^n$. For such a set E we have

$$B_E^p = \{0\}, \qquad 1 \leqslant p < 2,$$

by Theorem 2.5 and the Hausdorff–Young theorem.

Now we see that formulas (5) and (6) do not hold for the set E because $m(E) > 0$ but

$$\bar{K}_\varphi(B_E^p) = 0, \qquad 1 \leqslant p < 2.$$

5. The Case $2 < p \leqslant \infty$. More Counterexamples

In this section we prove that for $2 < p \leqslant \infty$ formulas (5) and (6) do not hold for some spectral sets of Lebesgue measure zero. It is clear that such a set cannot be closed, because by Remark 3.8 formulas (5) and (6) always hold for closed spectral sets of measure zero. It follows that formulas (5) and (6) are not true for some open spectral sets.

Theorem 5.1. *There exists a set $E \subset [-\pi, \pi]^n$ such that*

(a) $m(E) = 0$,

(b) $\bar{K}_\varepsilon(B_E^p) = 1$, $0 < \varepsilon \leqslant \varepsilon_p$, $2 < p \leqslant \infty$,

(c) $\bar{K}(B_E^p) = 1$, $2 < p \leqslant \infty$,

(d) $\bar{K}_\varphi(B_E^p) = 1$, $2 < p \leqslant \infty$,

for all functions φ satisfying conditions of Theorem 3.1 and such that $\varphi(t) \to 0$ as $t \to \infty$.

Proof. It is clear that (c) follows from (b). Moreover, we need only prove that there exists a spectral set $E \subset [\pi, \pi]^n$ of Lebesgue measure zero such that

$$\bar{K}_\varepsilon^{(l)}(B_E^p) \geqslant 1, \qquad 0 < \varepsilon < \varepsilon_p, 2 < p \leqslant \infty, \tag{37}$$

because we have, by Theorem 3.1,

$$\bar{K}_\varepsilon^{(u)}(B_E^p) \leqslant \bar{K}_\varphi(B_E^p) \leqslant \bar{K}_\varphi(B_{[-\pi, \pi]^n}^p) = 1.$$

It is known that there exists a singular measure μ over $[-\pi, \pi]$ which has closed support and satisfies

$$\mu[-\pi, \pi] = 1,$$

$$\sum_{k \in Z} |c_k(\mu)|^p < \infty, \qquad 2 < p < \infty,$$

where $c_k(\mu)$ denote the Fourier coefficients of the measure μ (see [33]).

Consider the Cartesian product of n copies of the measure μ and denote it by ν. This new measure satisfies

$$\sum_{k \in Z^n} |c_k(\nu)|^p < \infty, \qquad 2 < p < \infty. \tag{38}$$

It is clear that

$$m(\mathrm{supp}(\nu)) = 0.$$

Using dilations of measures it is easy to construct a measure ξ with support in $[-\pi, \pi]^n$ such that

$$|F\xi(x)| \geqslant \tfrac{1}{2}, \qquad x \in [-1, 1]^n. \tag{39}$$

The measure ξ equals some dilation of the measure ν. Moreover, we have

$$\xi([-\pi, \pi]^n) = 1.$$

By Theorem 2.5 and (38) it follows that

$$F\xi \in L^p(R^n), \qquad 2 < p < \infty. \tag{40}$$

Suppose j is a positive integer. Consider an n-dimensional net N_j of cubes in $[-\pi, \pi]^n$:

$$N_j = \{[k_1 \pi j^{-1}, (k_1 + 1)\pi j^{-1}] \times \cdots \times [k_n \pi j^{-1}, (k_n + 1)\pi j^{-1}]\},$$
$$-j \leqslant k_m < j, \ 1 \leqslant m \leqslant n.$$

For every cube $\Delta \in N_j$ defined by integers $k_1, ..., k_n$, consider the measure

$$\xi_\Delta(A) = \xi(j[(A \cap \Delta) - (k_1, ..., k_n)])$$

over the σ-algebra $\{A\}$ of all Borel subsets of $[-\pi, \pi]^n$. Set

$$E = \bigcup_j \bigcup_{\Delta \in N_j} \mathrm{supp}(\xi_\Delta).$$

We prove inequality (37) for this set E.

Consider a family of functions

$$l(x)=\sum_{\Delta\in N_j} a_\Delta F\xi_\Delta(x),$$

where the a_Δ are certain coefficients. It is clear that

$$l(x)=v(j^{-1}x)\,F\xi(j^{-1}x),$$

where

$$v(y)=\sum b_k e^{ikx} \tag{41}$$

and the sum in (41) is taken over all $k\in Z^n$ such that $|k_l|\leqslant j$, $1\leqslant l\leqslant n$. In (41), b_k denotes the coefficient a_Δ, where the cube Δ corresponds to $k=(k_1,...,k_n)$.

It follows by $2j$-periodicity of $v(j^{-1}x)$ that

$$\|l\|_p=\left\{j^n\int_{[-1,1]^n}|v(y)|^p\sum_{k\in Z^n}|F\xi(y+2k)|^p\,dy\right\}^{1/p}. \tag{42}$$

Using Theorem 2.5 and the measure λ_y, $y\in[-1,1]^n$, given by

$$d\lambda_y(x)=e^{-iyx}\,d\xi(x),$$

we get

$$\begin{aligned}\sum_{k\in Z^n}|F\xi(y+2k)|^p=\sum_{k\in Z^n}|F\lambda_y(2k)|^p&\leqslant\sum_{k\in Z^n}|F\lambda_y(k)|^p\\&\leqslant c_p\,\|F\lambda_y\|_p^p\leqslant c_p\,\|F\xi\|_p^p.\end{aligned}$$

Now (42) implies

$$\|l\|_p\leqslant c_p\,\|F\xi\|_p\left\{\int_{[-1,1]^n}|v(y)|^p\,dy\right\}^{1/p}j^{n/p}.$$

On the other hand, by (39) we have

$$\begin{aligned}\|l\|_{p,[-j,j]^n}&=\left\{j^n\int_{[-1,1]^n}|v(y)|^p\,|F\xi(y)|^p\,dy\right\}^{1/p}\\&\geqslant c_p j^{n/p}\left\{\int_{[-1,1]^n}|v(y)|^p\,dy\right\}^{1/p}.\end{aligned}$$

Thus

$$\|l\|_p \leqslant c_p \, \|F\xi\|_p \, \|l\|_{p,[-j,j]^n}.$$

This inequality shows that the class $B_E^p[-j,j]^n$ contains a $(2j+1)^n$-dimensional ball of radius $\tau_p = [c_p \, \|F\xi\|_p]^{-1}$. Using Tikhomirov's theorem on the m-widths of finite-dimensional balls (Theorem 2.3) and (40) we obtain the required estimate (37) with $\varepsilon_p = \tau_p$, $2 < p < \infty$. The case $p = \infty$ is similar.

It follows easily from Theorem 5.1 that the following assertion holds.

COROLLARY 5.2. *For every $\delta > 0$ there exists an open set $E \subset [-\pi, \pi]^n$ such that*

(a) $m(E) \leqslant \delta$,

(b) $\bar{K}_\varepsilon(B_E^p) = 1$, $0 < \varepsilon < \varepsilon_p$, $2 < p \leqslant \infty$,

(c) $\bar{K}(B_E^p) = 1$, $2 < p \leqslant \infty$,

(d) $K_\varphi(B_E^p) = 1$, $2 < p \leqslant \infty$,

for all functions φ satisfying conditions of Theorem 3.1 *and such that* $\varphi(t) \to 0$, $t \to \infty$.

6. THE CASES $p = 2$ AND $p = \infty$. TIKHOMIROV'S FORMULA IS TRUE FOR CLOSED SPECTRAL SETS

Suppose the spectral set E of a class B_E^p is closed. Then by the first part of Remark 3.8 the formula

$$\bar{K}_\varphi(B_E^p) = (2\pi)^{-n} \, m(E) \tag{43}$$

is true for all functions φ, satisfying conditions of Theorem 3.1 and the additional condition $\varphi(t) \leqslant \varepsilon_0$, as long as the lower estimate

$$\bar{K}_\varepsilon^{(l)}(B_E^p) \geqslant (2\pi)^{-n} \, m(E) \tag{44}$$

holds for $0 < \varepsilon < \varepsilon_0$. In this section we prove (44) for $p = 2$ and $p = \infty$.

THEOREM 6.1. *If the spectral set E of a class B_E^∞ is closed and if the function φ satisfies conditions of Theorem* 3.1, *then formula* (43) *holds.*

Proof. Without loss of generality we can assume $E \subset [-\pi, \pi]^n$.

As in Section 2 let us consider a net W consisting of all points $k \in R^n$ with integral coordinates, and the nets $W(r, x)$ with $r > 0$, $x \in R^n$. By Theorem 2.4 for every $\delta > 0$ there exists $r_\delta > 0$ and for every $r > r_\delta$ there exists $x_\delta \in R^n$ such that at least $(m(E)-\delta)\, r^{-n}$ points of the net $W(r, x_r)$ belong to E.

Choose k such that $k^{-1}\pi \leqslant r_\delta$. Then, using the ideas in the proof of Theorem 5.1, we can construct appropriate spaces of polynomials and show that the class $B_E^\infty[-k, k]^n$ contains an l-dimensional unit ball with

$$l \geqslant (m(E)-\delta)\, k^n \pi^{-n}.$$

It follows from this estimate and from Theorem 2.3 that

$$K_\varepsilon(B_E^\infty[-k, k]^n) \geqslant (m(E)-\delta)\, k^n \pi^{-n} - 1$$

and

$$\liminf_{k \to \infty} (2k)^{-n} K_\varepsilon(B_E^\infty[-k, k]^n) \geqslant (2\pi)^{-n}\, m(E).$$

This gives (44) and Theorem 6.1 follows.

THEOREM 6.2. *Suppose a spectral set E of a class B_E^2 is closed. Then there exists $\varepsilon_0 > 0$ such that inequality* (44) *holds for all numbers ε with $0 < \varepsilon < \varepsilon_0$. If the function φ satisfies conditions of Theorem* 3.1 *and the additional condition $\varphi(t) \leqslant \varepsilon_0$, $t > 0$, then formula* (43) *holds.*

Proof. As in Theorem 6.1, without loss of generality we assume $E \subset [-\pi, \pi]^n$.

Suppose $\theta > 0$ and $\lambda > 0$ are given. It is easy to show, using the Lebesgue density theorem and Egorov's theorem, that there exist a set $G \subset E$ and a positive number r_0 such that

$$m(E \backslash G) \leqslant \lambda, \tag{45}$$

$$m((x + r[0, 1]^n) \cap E) \geqslant (1-\theta)\, r^n, \qquad 0 < r < r_0,\ x \in G. \tag{46}$$

By Theorem 2.4, for every $\delta > 0$ there exists r_δ with $0 < r_\delta < r_0$ such that for every $r < r_\delta$ there exists $x_r \in R^n$ for which at least $(m(G)-\delta)\, r^{-n}$ points of the net $W(r, x_r)$ belong to G.

Now choose k such that $k^{-1}\pi < r_\delta$ and let $\{y_1, ..., y_l\}$ denote the points of the net $W(r, x_r)$ with $r = k^{-1}\pi$ belonging to G. As we mentioned above,

$$l \geqslant (m(G)-\delta)\, r^n. \tag{47}$$

Consider the class of functions

$$V=\{p:\ \|p\|_{2,[-k,k]^n}\leqslant 1\},$$

$$p(x)=\sum_{m=1}^{l} a_m F\chi_{A_m\cap E}(x), \qquad A_m=y_m+[0,k^{-1}\pi]^n,\ 1\leqslant m\leqslant l,$$

where the $\{a_m\}$ are certain coefficients.

Our goal is to estimate $\|p\|_2$, $p\in V$. By Plancherel's equality and (46) we have

$$1\geqslant \|p\|_{2,[-k,k]^n}\geqslant \|q\|_{2,[-k,k]^n}-c\theta^{1/2}\left\{\sum_{m=1}^{l}|a_m|^2\right\}^{1/2}k^{-n/2}, \tag{48}$$

where we set

$$q(x)=\sum_{m=1}^{l} a_m F\chi_{A_m}(x).$$

It is clear that

$$|q(x)|=|F\chi_{D_k}(x)|\ |z(x)|, \tag{49}$$

where

$$D_k=[0,k^{-1}\pi]^n,$$

and Z is a $2k$-periodic trigonometrical polynomial with coefficients a_k, $1\leqslant k\leqslant l$.

For $x\in[-k,k]^n$ we have

$$|F\chi_{D_k}(x)|=c\ |x_1\cdots x_n|^{-1}\ |\sin(2k)^{-1}\pi x_1|\cdots|\sin(2k)^{-1}\pi x_n|\geqslant ck^{-n}.$$

Hence, (49) yields

$$|q(x)|\geqslant ck^{-n}\ |z(x)|$$

and using Bessel's inequality we obtain

$$\|q\|_{2,[-k,k]^n}\geqslant ck^{-n}\ \|z\|_{2,[-k,k]^n}\geqslant ck^{-n/2}\left\{\sum_{m=1}^{l}|a_m|^2\right\}^{1/2}.$$

From this estimate and (48), we get, for small values of θ,

$$\left\{\sum_{m=1}^{l}|a_m|^2\right\}^{1/2}k^{-n/2}\leqslant c.$$

Fix such a number θ. Then Plancherel's equality yields

$$\|p\|_2 \leqslant c \left\{ \sum_{m=1}^{l} |a_m|^2 \right\}^{1/2} k^{-n/2} \leqslant c.$$

It follows that the class $B_E^2[-k, k]^n$ contains an l-dimensional ball of radius c^{-1}. By Theorem 2.3 and (47), we have

$$K_\varepsilon(B_E^2[-k, k]^n) \geqslant (m(G) - \delta)\, \pi^{-n} k^n, \qquad \varepsilon < c^{-1}.$$

Hence the lower estimate (44) holds for $\varepsilon < c^{-1}$, since (45) is true and the numbers λ and δ are arbitrary.

This completes the proof of Theorem 6.2.

7. Necessary Conditions for the Function φ in Tikhomirov's Formula

Let us consider the case $n = 1$, $E = [-\pi, \pi]$. By Theorem 3.1 we have

$$\bar{K}_\varphi(B^p_{[-\pi, \pi]}) = 1, \qquad 1 \leqslant p \leqslant \infty, \tag{50}$$

for every positive non-increasing function φ on $(0, \infty)$ such that

(1) $\varphi(t) < 1$, $t > 0$;

(2) $\int_0^\infty (|\ln \varphi(t)|/(1 + t^2))\, dt < \infty$.

In this section the following problem is discussed: what are exact conditions on the function φ under which formula (50) holds? The complete answer to this question is unknown. In what follows we give examples of functions φ for which formula (50) does not hold. It can be seen that the gap between the sufficient conditions (1, 2) and the examples given in Theorem 7.1 is not large.

THEOREM 7.1. *Assume*

$$\varphi_\varepsilon(t) = \exp\{-(1 + \varepsilon)\, t \ln(1 + t)\}, \qquad t > 0,\ \varepsilon > 0.$$

Then

$$\bar{K}^{(l)}_{\varphi_\varepsilon}(B^p_{[-\pi, \pi]}) \geqslant 1 + \varepsilon, \qquad 1 \leqslant p \leqslant \infty,$$

and formula (50) *does not hold in this case.*

Proof. Set $\Delta = [-\pi, \pi]$ and for every positive integer m consider the m-fold convolution

$$g_m = \underbrace{\chi_\Delta * \cdots * \chi_\Delta}_{m \text{ times}}.$$

Put

$$h_m(x) = (2m-1)\, g_m((2m-1)\, x), \qquad x \in R^1, m \geqslant 2.$$

It follows that $\operatorname{supp}(h_m) \subset \Delta$ and

$$Fh_m(x) = (\pi x)^{-m} (2m-1)^m \left(\sin \frac{\pi x}{2m-1} \right)^m, \qquad x \in R^1, m \geqslant 2.$$

Fix $k \geqslant 2$ and consider a family of functions

$$V = \{q(x)\, Fh_l(x), x \in R^1, l = [(2+\varepsilon)\, k]\},$$

where

$$q(x) = \sum_{j=0}^{l-2} a_j x^j, \qquad x \in R^1,$$

and $[y]$ denotes the integral part of y.

Suppose $p = \infty$. The next step in the proof of Theorem 7.1 consists of estimating $\|v\|_\infty$, $v \in V$, provided

$$|v(x)| \leqslant 1, \qquad x \in [-k, k]. \tag{51}$$

We need the following known theorem.

THEOREM 7.2 (S. Bernstein, see [1, p. 323]). *Suppose a polynomial*

$$q(x) = \sum_{j=0}^{m} a_j x^j$$

satisfies

$$|q(x)| \leqslant M, \qquad -k \leqslant x \leqslant k.$$

Then

$$|a_j| \leqslant \frac{M}{(2[j/2])!} \left(\frac{m}{k} \right)^j, \qquad 0 \leqslant j \leqslant m.$$

If $v \in V$ satisfies (51) then for

$$q(x) = (Fh_l(x))^{-1}\, v(x)$$

we have

$$|q(x)| \leqslant (\pi\, |x|)^l\, (2l-1)^{-l} \left| \sin \frac{\pi x}{2l-1} \right|^{-l} \leqslant \left(\frac{\pi}{2} \right)^l, \qquad -k \leqslant x \leqslant k. \tag{52}$$

Now Theorem 7.2 and (52) yield

$$|a_j| \leqslant \left(\frac{\pi}{2}\right)^l \frac{(2+\varepsilon)^j}{(2[j/2])!}, \qquad 0 \leqslant j \leqslant l-2. \tag{53}$$

Hence, for $|x| > k$ and $\alpha = [(1+\varepsilon/2)k]$ we have, by (53), that

$$\begin{aligned} |q(x)| &\leqslant \left| \sum_{j=0}^{\alpha} a_j x^j \right| + \left| \sum_{j=\alpha+1}^{l-2} a_j x^j \right| \\ &\leqslant \left(\frac{\pi}{2}\right)^l (2+\varepsilon)^\alpha |x|^\alpha + \left(\frac{\pi}{2}\right)^l \frac{(2+\varepsilon)^l}{(\alpha-1)!} |x|^{l-2}. \end{aligned}$$

Therefore,

$$|v(x)| \leqslant (2l-1)^l (\pi |x|)^{-l} |q(x)| \leqslant l^l \left((2+\varepsilon)^\alpha k^{\alpha-l} + \frac{(2+\varepsilon)^l}{(\alpha-1)!} \right).$$

Since

$$m! \geqslant m^m e^{-m+1},$$

we have

$$|v(x)| \leqslant \beta^{\delta k} k^{(1+\varepsilon/2)k}, \qquad |x| > k,\ k > k_0, \tag{54}$$

where β, δ, and k_0 are absolute positive constants.

It follows from (51) and (54) that, for $k > k_0$, the class $B_\Delta^\infty[-k, k]$ contains an l-dimensional ball of radius

$$r_k = \beta^{-\delta k} k^{-(1+\varepsilon/2)k}.$$

For every $\tau > 0$ there exists $k_1 > k_0$ depending only on τ such that

$$\varphi_{\lambda+\tau}(k) \leqslant r_k, \qquad k > k_1,\ \lambda = \varepsilon/2.$$

Hence by Theorem 2.3

$$K_{\varphi_{\lambda+\tau}}(B_\Delta^\infty[-k, k]) \geqslant l \geqslant (2+\varepsilon)k - 1, \qquad k > k_1,$$

and

$$\bar{K}^{(l)}_{\varphi_{\lambda+\tau}}(B_\Delta^\infty) \geqslant 1 + \lambda, \qquad \tau > 0.$$

Theorem 7.1 in the case $p = \infty$ follows from this inequality.

When $1 \leqslant p < \infty$ the proof is similar. First, suppose $v \in V$ satisfies

$$\|v\|_{p, [-k, k]} \leqslant 1.$$

Then

$$|q(x)| \leqslant \left(\frac{\pi}{2}\right)^{l} |v(x)|, \qquad -k \leqslant x \leqslant k,$$

and

$$\|q\|_{1,\,[-k,\,k]} \leqslant \left(\frac{\pi}{2}\right)^{l} \int_{-k}^{k} |v(x)|\, dx \leqslant \left(\frac{\pi}{2}\right)^{l} (2k)^{1/p'} \|v\|_{p,\,[-k,\,k]}$$

$$\leqslant \left(\frac{\pi}{2}\right)^{l} (2k)^{1/p'}, \qquad \frac{1}{p}+\frac{1}{p'}=1.$$

Define a new polynomial

$$z(x)=\sum_{j=0}^{l-2} (j+1)^{-1} a_j x^{j+1}.$$

It is clear that $q(x)=z'(x)$, and

$$|z(x)|=|z(x)-z(0)| \leqslant \int_{-k}^{k} |z'(u)|\, du=\int_{-k}^{k} |q(u)|\, du$$

$$\leqslant (2k)^{1/p'} \left(\frac{\pi}{2}\right)^{l}, \qquad -k \leqslant x \leqslant k.$$

By Theorem 7.2

$$|a_j| \leqslant (j+1)(2k)^{1/p'} \left(\frac{\pi}{2}\right)^{l} \frac{(2+\varepsilon)^j}{(2[j/2])!}, \qquad 0 \leqslant j \leqslant l-2.$$

Moreover, reasoning as above we obtain

$$|v(x)| \leqslant \beta^{\delta k} l^{l} |x|^{-l} (|x|^{\alpha}+\alpha^{-\alpha} |x|^{l-2}), \qquad -k \leqslant x \leqslant k,\ k>k_0,$$

where β, δ, and k_0 are some positive constants depending only on p.
Put $\tau=(l-\alpha)p-1$. Thus we have

$$\left\{\int_{|x|>k} |v(x)|^{p}\, dx\right\}^{1/p} \leqslant \beta^{\delta k} l^{l} \{(\tau^{-1} k^{-\tau})^{1/p}+\alpha^{-\alpha}(2p-1)^{1/p} k^{2+1/p}\}$$

$$\leqslant \rho^{\kappa k} k^{(1+\varepsilon/2)k}, \qquad k>k_2,$$

where the positive constants ρ, κ, and k_2 depend only on p.
Now the proof of Theorem 7.1 in the case $1 \leqslant p<\infty$ can be completed along the same lines as the proof for $p=\infty$.

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